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# Divergence among estimators of the population mean under right-skewed distributions and outliers – a case study of metal accumulation in Cu-Ag deposits of the LGCD

## Introduction

The principal parameters used in the evaluation of metal ore deposits include grade (or metal content), which expresses the concentration of a useful component in the ore (e.g., g/t, wt%). Another key parameter is metal accumulation, which represents the mass of metal per unit area of the deposit (e.g., $kg/m^2$, $g/m^2$). Metal accumulation is calculated as the product of grade and thickness, and, when density variations occur, as grade × thickness × density (David 1977; Journel and Huijbregts 1978; Annels 1991; Glacken and Snowden 2001; Sinclair and Blackwell 2002; Bertoli et al. 2003). The grade and metal accumulation typically exhibit strongly right-skewed empirical probability distributions that

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contain numerous outliers, sometimes referred to as anomalously high or extreme values. This results in significant challenges related to:

- statistically satisfactory approximation of empirical distributions by theoretical models,
- accurate estimation of metal resources according to accepted criteria,
- proper assessment of ore quality, usually expressed as mean metal grades (including useful and deleterious components).

For distributions characterized by strong positive skewness (with a long tail toward high values), the statistical literature recommends, along with the arithmetic mean, the use of alternative measures of central tendency such as the median, geometric mean, trimmed mean, or Winsorized mean (David 1977; Journel and Huijbregts 1978; Rock 1988; Sinclair and Blackwell 2002).

According to statistical theory, for ideal theoretical right-skewed distributions, these measures (with the exception of the arithmetic mean) constitute biased estimators of the expected value. Assuming hypothetically that the probability distribution of the metal accumulation in the deposit is known, the expected value can be calculated as follows:

$$E(q) = \int_a^b q \cdot f(q)dq \tag{1}$$

↳ $q$ – accumulation of metals,
$f(q)$ – probability density function of metal accumulation,
$a$, $b$ – lower and upper bounds of $q$.

The expected value $E(q)$, multiplied by the deposit area $F$, yields the metal resources $Q$:

$$Q = E(q) \cdot F \tag{2}$$

This theoretical approach to the estimation of metal resources is rarely applicable in practice. In geological and mining applications, the theoretical distributions of metal accumulation are unknown; only approximations of empirical distributions obtained on the basis of the results of deposit sampling using theoretical distributions are available. The use of theoretical distributions is justified only when appropriate statistical tests do not reject the hypothesis that they are consistent with the empirical distribution. Numerous outliers in the empirical datasets often lead to the rejection of this hypothesis, as demonstrated in Section 3 for the accumulation of the metals analyzed in the Cu-Ag deposit. Another consequence of the occurrence of outliers is uncertainty regarding which measure of central tendency provides estimates closest to the reference population mean. For this reason, the present study focuses on quantifying the divergence among estimates of the population mean of metal accumulation obtained using different measures of central tendency, with the population mean regarded as the parameter relevant to additive resource estimation, rather than on identifying a universally preferred descriptive measure for skewed datasets.

The expected value of metal accumulation can theoretically be understood as the mean value of metal accumulation across the deposit, which, however, cannot be determined directly in practice and can only be approximated from sampling data. Verification of the evaluation of the expected value using measures of central tendency calculated from sampling results is also impossible, even after the completion of mining operations, due to unavoidable dilution by barren rock and sub-economic ore, as well as the necessity of leaving part of the ore in situ to ensure worker safety in the event of unexpected natural hazards.

In applied ore quality assessment, the median and, less frequently, the geometric mean are sometimes used as alternative measures of central tendency for variables with a right-skewed distribution. Such measures may be useful for descriptive purposes or for robust characterization of strongly asymmetric datasets. However, their usefulness as estimators of the population mean relevant to resource evaluation remains uncertain and requires empirical verification. Concerns about the arithmetic mean usually arise from its sensitivity to extreme values and from the possibility of overestimating mean accumulation and, consequently, the resource quantity.

The metal resource estimate can be obtained from the formula:

$$Q_{AM(q)} = \overline{q} \cdot F \tag{3}$$

↳ $\overline{q}$ – the arithmetic mean of the accumulation of metal,
$F$ – the deposit area.

If the median or the geometric mean were shown to provide a better approximation of the population mean relevant to resource estimation, the resource estimation could be modified accordingly by replacing the average-based estimate ($Q_{AM}$) with a median-based estimate ($Q_{Me}$) or a geometric-mean-based estimate ($Q_{GM}$), using formulas:

$$Q_{Me(q)} = Me \cdot F \tag{4}$$

or

$$Q_{GM(q)} = q_{GM} \cdot F \tag{5}$$

Accordingly, the aim of this study is not to identify a universally preferred measure of central tendency for skewed data, but to assess the performance of selected estimators in approximating the population mean relevant to additive resource estimation.

## 1. Research objective

The objective of this study was to quantify the divergence among estimates of the population mean of metal accumulation, relevant to additive resource estimation, obtained using different measures of central tendency used as alternative estimators of that target parameter under conditions of strong positive skewness (right-skewed distributions) and numerous outliers. For this purpose, five measures of central tendency were used: the arithmetic mean, the median, the geometric mean, the trimmed mean and the Winsorized mean. Their values were calculated for random samples of five different sizes, generated by the Monte Carlo method from empirical distributions. The analyses were carried out using measurements of the accumulation of four metals (Cu, Co, Ni, Pb) occurring in the Cu-Ag deposits of the LGCD area (Piestrzyński ed. 1996), selected as representative examples of the statistical conditions considered in this study.

## 2. Research material and data statistics

A large dataset was compiled from sampling results obtained in the underground mine workings of the Cu-Ag Rudna deposit, operated by KGHM Polska Miedź S.A. It consisted of the accumulation of four metals: the main metal (Cu) and three accompanying elements (Co, Ni, and Pb), determined within the vertical boundaries of the ore deposit. The accumulation of these elements is defined as their mass (expressed in kilograms or grams) per unit area of 1 $m^2$ of the deposit in the horizontal plane. Two of the accompanying elements, nickel (Ni) and lead (Pb), are partially recovered during the ore processing operations. The accumulation values of the analyzed metals are characterized by strongly right-skewed distributions and numerous outliers, whereas the Ni dataset contains markedly fewer such extreme observations. A comprehensive geological description of the deposit and the sampling procedure is presented in Auguścik-Górajek et al. (2021). The basic statistics of the datasets used in the research are provided in Table 1.

The median (*Me*), geometric mean (*GM*), trimmed mean (*TM*) and Winsorized mean (*WM*) are presented in Table 1 for illustrative purposes only, to demonstrate the divergence between commonly used measures of central tendency for the same empirical data under strong right skewness and the presence of outliers. These measures are not used for resource estimation, which requires additive properties fulfilled only by the arithmetic mean. From a practical geological perspective, the values shown in Table 1 illustrate the potential scale of systematic underestimation of metal resources that may result from replacing the arithmetic mean with alternative measures of central tendency.

As shown in Table 1, the datasets vary substantially in size, although all are large, especially those for Cu and Pb. The differing numbers of observations reflect differences in the average horizontal spacing between sampling points at which metal contents were determined (20–40 m for Cu, 180 m for Pb, and approximately 400 m for Co and Ni).

Table 1. Basic statistics of metal accumulations ($q$) calculated for full empirical data sets

Tabela 1. Podstawowe statystyki zasobności metali ($q$) obliczone dla pełnych zbiorów danych

| Metal accumulation | $N$ | $\overline{q}$ | $Me$ | $GM$ | $TM$ | $WM$ | $v$ (%) | $g_1$ |
|---|---|---|---|---|---|---|---|---|
| $qCu$ (kg/m$^2$) | 10,243 | 272.33 | 228.56 | 225.22 | 235.43 | 241.98 | 68 | 2.27 |
| $qCo$ (g/m$^2$) | 732 | 426.61 | 331.70 | 296.51 | 338.40 | 346.48 | 96 | 2.88 |
| $qNi$ (g/m$^2$) | 722 | 391.12 | 338.40 | 316.20 | 349.20 | 358.26 | 65 | 1.56 |
| $qPb$ (kg/m$^2$) | 3,255 | 7.75 | 2.33 | 2.60 | 3.27 | 4.33 | 166 | 2.81 |

$N$ – size of the data set, $\overline{q}$ – arithmetic mean, $Me$ – median, $GM$ – geometric mean, $TM$ – trimmed mean, $WM$ – Winsorized mean, $v$ (%) – coefficient of variation, $g_1$ – coefficient of skewness.

The accumulation of Cu, Co, and Ni exhibits high variability, with coefficients of variation ($v$) ranging from 65% to 96%, whereas Pb shows extremely high variability ($v$ = 166%). The high values of the skewness coefficients ($g_1$), from 1.56 ($qNi$) to 2.88 ($qCo$), indicate strong to very strong positive skewness (right-skewed distributions) in the empirical data.

The quantitative statistical characteristics are fully reflected in the plots of their empirical distributions (histograms) and in the theoretical distributions fitted to them according to certain criteria (solid lines), as well as in box plots shown in Figure 1. The box plots provide a clearer characterization of the data points conventionally classified as outliers, which are marked in red. These include all observations greater than the upper quartile plus three times the interquartile range. A review of the plots reveals only three outliers in the empirical distribution of Ni accumulation, numerous outliers for Cu and Co accumulation, and a very large number of outliers in the empirical distribution of Pb accumulation.

The selection of the best-fitting theoretical distributions was based on two criteria: goodness of fit assessed in STATGRAPHICS and the assumption that the probability density function is defined for non-negative variable values (STATGRAPHICS software: Statgraphics Technologies, Inc. 2024).

A preliminary visual comparison of the theoretical and empirical distributions suggests that the fit is close for Cu accumulation and reasonably satisfactory for the remaining elements.

The goodness of fit of theoretical distributions to empirical ones was verified using 7 goodness-of-fit tests available in STATGRAPHICS software (Statgraphics Technologies, Inc. 2020) at a significance level of 0.05. The test results summarized in Table 2 indicate that unambiguous conclusions are possible only for Pb and Ni accumulation. In the case of Ni accumulation, all tests indicate no grounds for rejecting the hypothesis that it follows a gamma distribution in the general population, which supports the use of this distribution as a working approximation in the present study, whereas in the case of Pb accumulation,

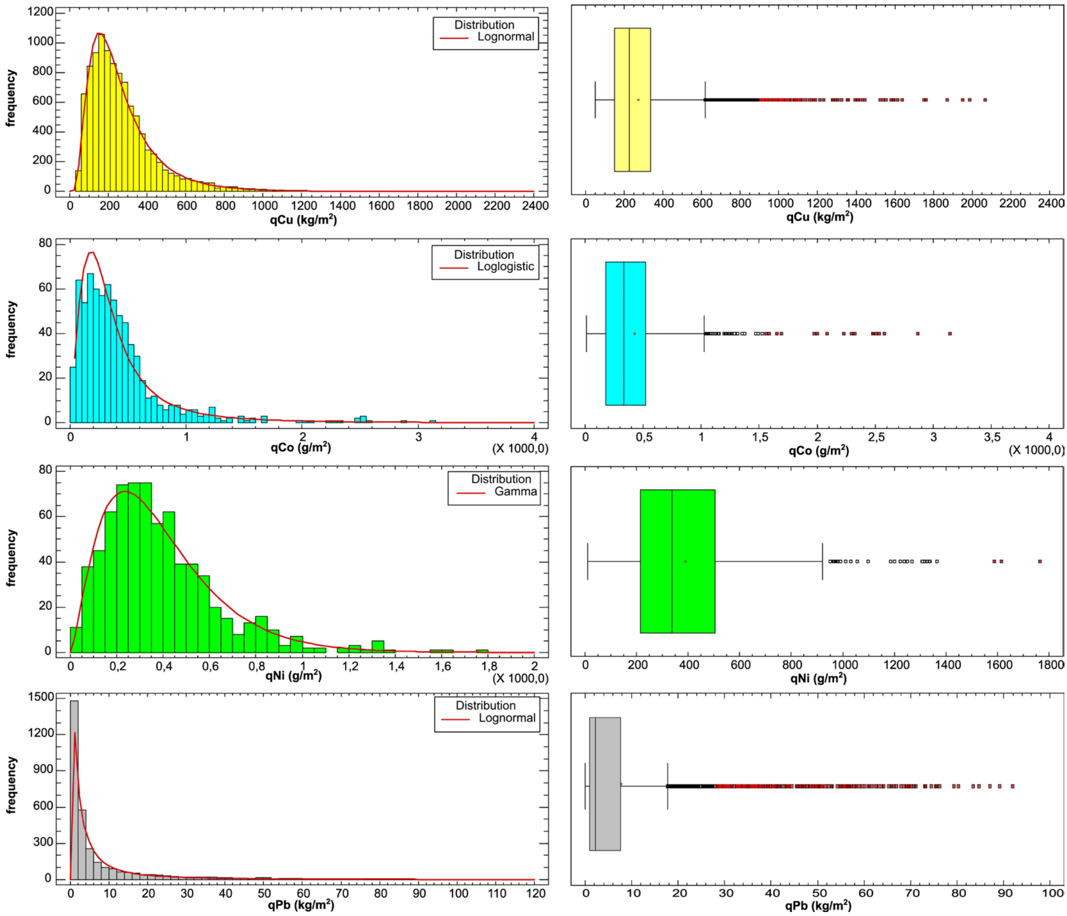


Fig. 1. Empirical distributions and best-fitting theoretical distributions (left) and box plots (right) for the accumulation of the analyzed metals

Rys. 1. Rozkłady empiryczne i najlepiej dopasowane rozkłady teoretyczne (z lewej) oraz wykresy pudełkowe (z prawej) zasobności rozpatrywanych pierwiastków

all tests consistently reject the hypothesis that this resource parameter follows a lognormal distribution in the general population.

For Cu and Co accumulation, the results are inconclusive, as only three out of seven statistical tests do not provide sufficient evidence to reject the hypotheses that their distributions in the general populations are lognormal and log-logistic, respectively, whereas the remaining four tests clearly reject these assumptions. Given the inconclusive and partly contradictory outcomes of the statistical tests, accepting the proposed theoretical distribution forms for the accumulation of Cu and Co in the general population must be regarded as highly uncertain and methodologically questionable.

Table 2. Goodness-of-fit tests of the "best" theoretical distributions to the empirical distributions of metal accumulation for the significance level α = 0.05

Tabela 2. Testy zgodności „najlepszych" rozkładów teoretycznych z rozkładami empirycznymi zasobności metali dla poziomu istotności α = 0,05

| Goodness-of-fit tests | *qCu* (kg/m²) Lognormal | *qCo* (g/m²) Loglogistic | *qNi* (g/m²) Gamma | *qPb* (kg/m²) Lognormal |
|---|---|---|---|---|
| | *P-value* | | | |
| Chi-Squared | <0.010 | <0.010 | 0.075 | <0.01 |
| Kolmogorov-Smirnov D | 0.155 | 0.187 | 0.614 | <0.01 |
| Modified Kolmogorov-Smirnov D | ≥0.100 | ≥0.100 | ≥0.100 | <0.01 |
| Kuiper V | <0.010 | <0.010 | ≥0.100 | <0.01 |
| Cramer-Von Mises $W^2$ | 0.05–0.10 | 0.05–0.10 | ≥0.100 | <0.01 |
| Watson $U^2$ | <0.010 | <0.010 | ≥0.100 | <0.01 |
| Anderson-Darling $A^2$ | 0.01–0.05 | 0.01–0.05 | ≥0.100 | <0.01 |

*P*-value > 0.05 – there is no basis for rejecting the null hypothesis of the assumed distribution; otherwise, the null hypothesis should be rejected.

Only for Ni and Pb accumulation can the results of the goodness-of-fit tests be regarded as fully unambiguous (Table 2). In the case of Ni, it is justified to adopt the gamma distribution as a working hypothesis for this variable, which enables subsequent analyses to incorporate random samples drawn not only from the empirical distribution but also from its fitted theoretical distribution (Table 2).

## 3. Research methodology

To assess the performance of several estimators under conditions typical of geological practice, a simulation scheme based on the Monte Carlo method was applied (Metropolis and Ulam 1949; Rubinstein and Kroese 2016). Estimator performance was assessed using 1,000 independent Monte Carlo realizations generated for each metal and for each sample size ($n \in \{10, 20, 50, 100, 200\}$) from empirical distributions (based on the complete dataset). Thus, for each metal and each sample size, 1000 independent random samples were drawn without replacement from the corresponding empirical dataset and used to calculate the five estimators compared in this study.

The complete empirical sets of data (Table 1) were treated as reference populations, with their arithmetic means adopted as reference approximations of the population mean relevant to additive resource estimation.

According to the goodness-of-fit results presented in Table 2, the generation of samples from theoretical distributions was limited exclusively to Ni, for which the gamma distribution was adopted as a working hypothesis. Thus:

- for Cu, Co, Ni and Pb, samples were generated from empirical distributions,
- for Ni, additional random samples were generated from the fitted gamma distribution.

Different sample sizes were considered to assess the robustness of individual estimators under conditions representative of geological practice, where data availability is frequently constrained. For each realization of a random sample, five measures were computed: arithmetic mean (*AM*), median (*Me*), geometric mean (*GM*), trimmed mean (*TM*), and Winsorized mean (*WM*).

The formulas and method for calculating these measures, additionally illustrated with an example (a set of five observations), are presented in Table 3.

To compare estimator accuracy with respect to the reference population mean (μ), defined here as the arithmetic mean of the corresponding reference population, for each realization of a random sample *Sj* (where $j = 1, \ldots, N$ and $N = 1000$) and each measure $M_k$ (where $k \in \{AM, Me, GM, TM, WM\}$), the deviation from the reference population mean was determined as:

$$d_{jk} = M_k(S_j) - \mu \tag{6}$$

Next, for each sample *Sj*, ranks from 1 to 5 were assigned by ordering the values $|d_{jk}|$ in ascending order:

$$rank(M_k(S_j)) = \text{the position of } |d_{jk}| \text{ in the ascending order} \tag{7}$$

The measure with rank 1 is, in a given sample, the closest to μ, and the one with rank 5 is the furthest. The percentage of cases in which a given measure of central tendency is the closest to the reference population mean (rank 1) was calculated according to the following relationship:

$$P_k = \frac{L_k}{N} \cdot 100\% \tag{8}$$

↳ $L_k$ – denotes the number of random samples in which the measure $M_k$ received rank 1, $N = 1000$ is the total number of simulated random samples in the series.

The values of $P_k$ are shown in Figure 2.

To assess the average deviation of the estimators from the reference population mean μ, the mean relative difference between each estimator and μ was calculated:

$$\Delta_k = \frac{1}{N}\sum_{j=1}^{N}\frac{M_k\left(S_j\right)-\mu}{\mu}\cdot 100\% \tag{9}$$

The values of $\Delta_k$ are shown in Figure 3.

Table 3. Measures of central tendency: definitions and numerical example

Tabela 3. Miary tendencji centralnej: definicje i przykład numeryczny

| Measures of central tendency | Formula | Example for: $x_i$ = 10, 40, 50, 75, 125 |
|---|---|---|
| Arithmetic Mean (AM) | $\bar{X}_A = \frac{1}{n}\sum_{i=1}^{n} x_i$ , where $n$ is the sample size. | $\bar{X}_A$ = 1/5 (10 + 40 + 50 + 75 + 125) = 60 |
| Median (Me) | For data ordered in increasing order $x_{(1)} \le \ldots \le x_{(n)}$: $\text{Me} = \begin{cases} \frac{x_{n+1}}{2} & \text{when } n \text{ is odd} \\ \frac{1}{2}\left(x_{\left(\frac{n}{2}\right)} + x_{\left(\frac{n}{2}+1\right)}\right) & \text{when } n \text{ is even} \end{cases}$ where $x_{(i)}$ denotes the $i$-th order statistic. | Me = 50 |
| Geometric Mean (GM) | $\bar{X}_G = \sqrt[n]{x_1 \cdot x_2 \cdot \ldots \cdot x_n} = \left(\prod_{i=1}^{n} x_i\right)^{1/n}$ | $\bar{X}_G = \sqrt[5]{10 \cdot 40 \cdot 50 \cdot 75 \cdot 125} = 45.14$ |
| Trimmed Mean (TM) | $\bar{X}_T = \frac{1}{n-2k}\sum_{i=k+1}^{n-k} x_{(i)}$ where $k$ denotes the number of observations trimmed from each end and is calculated as $k = \alpha \cdot n$ (rounded to the nearest integer); α denotes the trimming fraction, interpreted as the proportion of data removed symmetrically from both tails. | For $\alpha$ = 20% and $n = 5 \Rightarrow k = 1$ $\bar{X}_T$ = 1/3 (~~10~~ + 40 + 50 + 75 + ~~125~~) = 55 |
| Winsorized Mean (WM) | Replace the $k$ smallest values with $x_{(k+1)}$ and the $k$ largest with $x_{(n-k)}$; then $\bar{X}_W = \frac{1}{n}\sum_{i=1}^{n} x_i^{(w)}$ where $k$ denotes the count modified at each tail and is computed as $k = \alpha \cdot n$ (rounded); α denotes the winsorizing fraction; $x_i^{(w)}$ denotes the winsorized values. | For $\alpha$ = 20% and $n = 5 \Rightarrow k = 1$ $\bar{X}_W$ = 1/5 (~~10~~ + 40 + 40 + 50 + 75 + 75 + ~~125~~) = 56 |

## 4. Research results

The results of the simulations of random samples of metal accumulation (Cu, Co, Ni, Pb) generated from empirical distributions (complete data sets) are summarized in Figures 2 and 3.

Together, these results provide complementary information on estimator performance: the frequency with which each estimator was closest to the reference population mean and the average directional bias of each estimator relative to that value.

For all metals and all sample sizes considered, the arithmetic mean yielded the highest number of estimates closest to the reference population mean (i.e. with rank no. 1) (Figure 2).

The advantage of the arithmetic mean over the remaining estimators of the mean value clearly increases with the growth of the size of individual random samples. For example, for Ni accumulation, the share of first ranks of the arithmetic mean for samples with the smallest size (10 observations) is 38% and increases almost twofold up to 75.4% for samples with the largest size (200 observations) (Figure 2). In general, in 80% of cases the share of first positions (rank 1) of this estimator exceeds 50%.

The most significant advantage of the arithmetic mean is recorded for Pb accumulation, which is characterized by the largest number of outliers, very strong skewness and high variability of Pb accumulation values (Table 1, Figure 1, Figure 2), and the smallest, though still large, advantage for Ni accumulation, which is characterized by the fewest outliers and the lowest skewness (Figure 1, Figure 2).

The Winsorized mean also ranks highly, which, thanks to its reduced sensitivity to extreme values, generally places second among the estimators, except for random samples with the smallest size (10 observations).

The median, which always yields estimates closer to the reference population mean than the geometric mean, may have limited practical utility, mainly for small samples (containing 10 or 20 observations). However, even in the most favorable case, its share of first-rank cases reaches a maximum of 21% (qNi). This value is almost twice as low as the corresponding share for the arithmetic mean (38.0%) (Figure 2). This can be explained by the fact that the empirical distribution of Ni accumulation has the smallest skewness and contains only single outliers, and is therefore, compared to the accumulation distributions of other metals, closest to a symmetric distribution.

Overall, across all cases considered (Figure 2), the closest approximation of the reference population means, defined here as the expected value relevant to additive resource estimation, is provided by the arithmetic mean and, secondly, by the Winsorized mean, while the worst are the median and the geometric mean. The trimmed mean generally occupies an intermediate position between these two groups of estimators.

Due to the random nature of generating values from empirical distributions, the position of particular estimators (apart from the arithmetic mean) for small numbers of observations may change when new series of random samples are generated, since, in general, their percentage shares in the ranking of first places do not differ drastically.

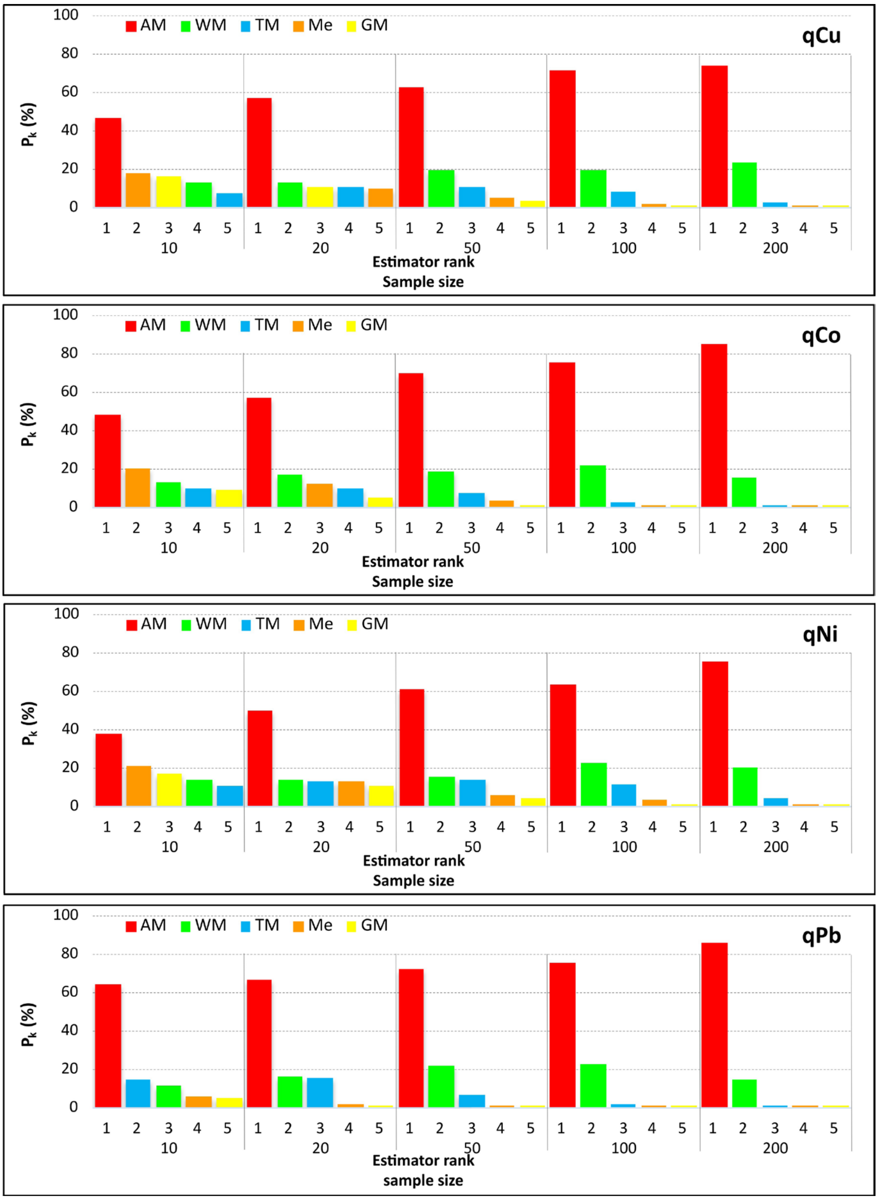


Fig. 2. Ranking of estimators with respect to their proximity to the reference population mean of metal accumulation (*q*) based on 1000 series of random samples of different sizes generated from empirical distributions *AM* – Arithmetic Mean, *Me* – Median, *GM* – Geometric Mean, *TM* – Trimmed Mean, and *WM* – Winsorized Mean; $P_k$ [%] – percentage share of estimators with values closest to the reference population mean; estimator rank (1–5): 1 – the estimate closest to the reference population mean, 5 – the estimate furthest from the reference population mean; sample size: 10, 20, 50, 100, 200 observations

Rys. 2. Ranking estymatorów pod względem bliskości oszacowań do referencyjnej średniej populacji zasobności metali (*q*), oparty na 1000 seriach losowych prób o zróżnicowanej liczebności, generowanych z rozkładów empirycznych

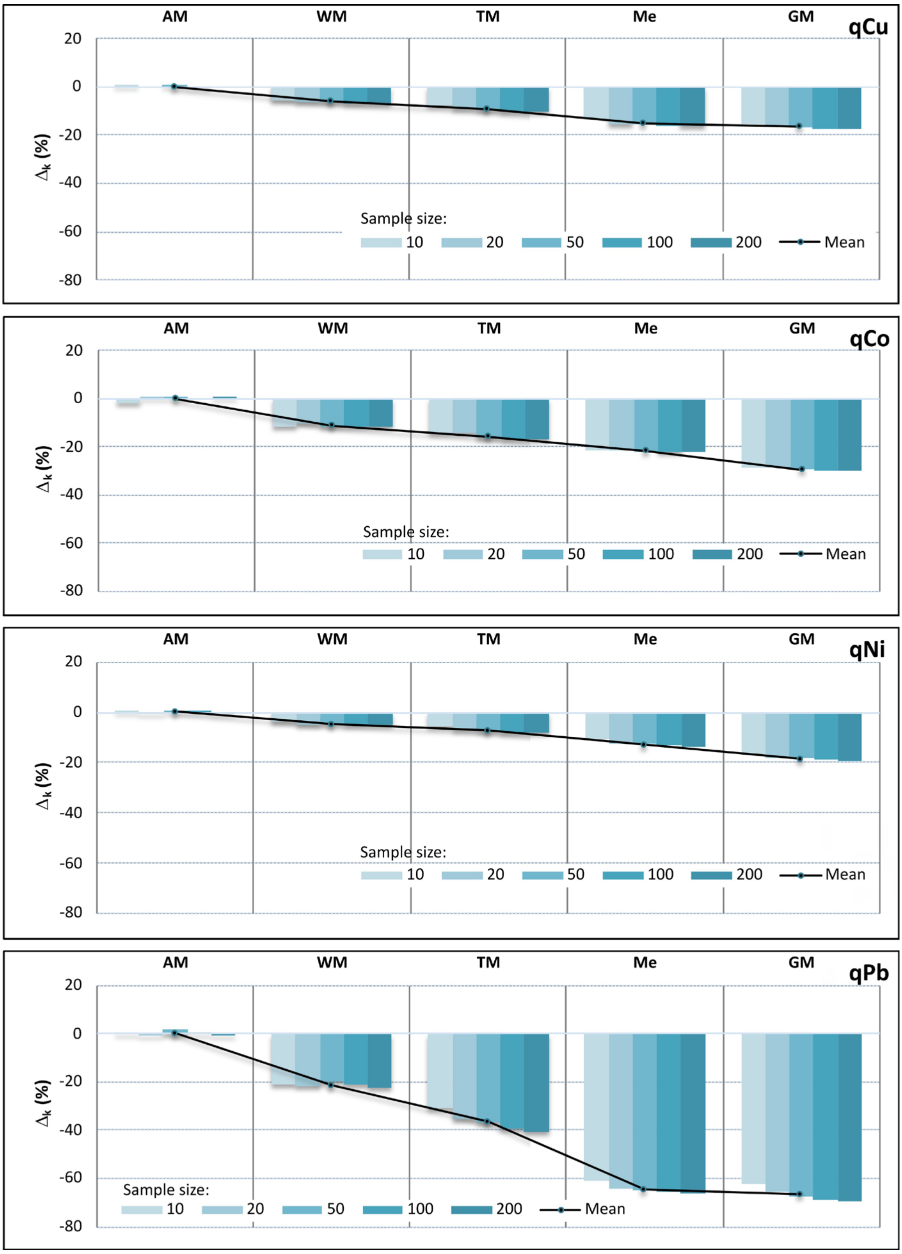


Fig. 3. Mean relative differences ($\Delta_k$) between the estimators (*AM*, *WM*, *TM*, *Me*, *GM*) and the reference population mean of metal accumulation (*q*) calculated on the basis of 1000 series of random samples of sizes: $n \in \{10, 20, 50, 100, 200\}$ generated from empirical distributions of metal accumulation
*AM* – Arithmetic Mean, *Me* – Median, *GM* – Geometric Mean, *TM* – Trimmed Mean, *WM* – Winsorized Mean

Rys. 3. Średnie względne różnice ($\Delta_k$) pomiędzy estymatorami (*AM*, *WM*, *TM*, *Me*, *GM*) a referencyjną średnią populacji zasobności metali (*q*), obliczone na podstawie 1000 serii losowych prób o liczebnościach: $n \in \{10, 20, 50, 100, 200\}$, generowanych z empirycznych rozkładów zasobności metali

In contrast to the ranking of measures of central tendency based on the number of results closest to the reference population mean, the ranking based on the mean difference between the estimates and the reference population mean provides unambiguous results (Figure 3), which makes it a clearer basis for comparing the average directional bias of the estimators. According to statistical theory, under right-skewed distributions, the arithmetic mean is an unbiased estimator of the expected value. Despite the numerous outliers, this is confirmed by the results shown in Figure 3, where the mean relative differences are very close to zero, deviating only slightly above or below this value. This bias increases consistently for the remaining measures, in the following order: winsorized mean, trimmed mean, median, and geometric mean. The degree of underestimation of the reference population mean depends on the variability and skewness of the empirical distributions of the four metal accumulations considered (Table 1). The average relative deviations from the reference population mean are smallest for the arithmetic mean and largest for the geometric mean. After averaging the results for random samples of all considered sizes (10, 20, 50, 100, and 200 observations), the deviations for both of these estimators are, respectively: for qCu: 0.0% and −16.6%, for qCo: −0.1% and −29.4%, for qNi: 0.1% and −18.3%, and for qPb: 0.4% and −66.7% (Figure 3). As shown in Figure 3, using the geometric mean (and, to a lesser extent, the median) as estimators of mean values leads to substantial, and sometimes very severe, underestimation of the reference population mean of metal accumulation and, consequently, to marked underestimation of resources when compared with estimates based on the arithmetic mean of metal accumulation.

A clearly smaller degree of underestimation is observed when using the trimmed mean and the Winsorized mean. The observed ranking of estimators reflects a trade-off between robustness and bias: estimators that are more resistant to outliers exhibit greater underestimation of the expected value.

The results of all 7 goodness-of-fit tests were unambiguous only for Ni accumulation (*qNi*) and did not provide grounds to reject the hypothesis that this parameter follows a gamma distribution (Table 2). For this reason, the procedure of ranking the estimators with respect to the reference population mean was repeated, based on a series of 1000 random samples, but this time generated from the theoretical gamma distribution (with parameters: shape = 2.505 and scale = 0.006). The results obtained were compared with the results previously obtained when generating random samples from the empirical distribution of Ni accumulation (Figure 4).

As shown in Figure 4, the arithmetic mean for random samples generated both from the theoretical distribution and from the empirical distribution most often yields estimates closest to the reference population mean. Compared to samples generated from the empirical distribution, the percentages associated with the individual ranks changed slightly for random samples of size 20 observations and to a lesser extent for samples of size 50 observations, although the overall ordering of the estimators remained essentially the same. However, the differences in the share of the highest rank of these measures are small. The average relative differences between the five measures of central tendency and the reference population

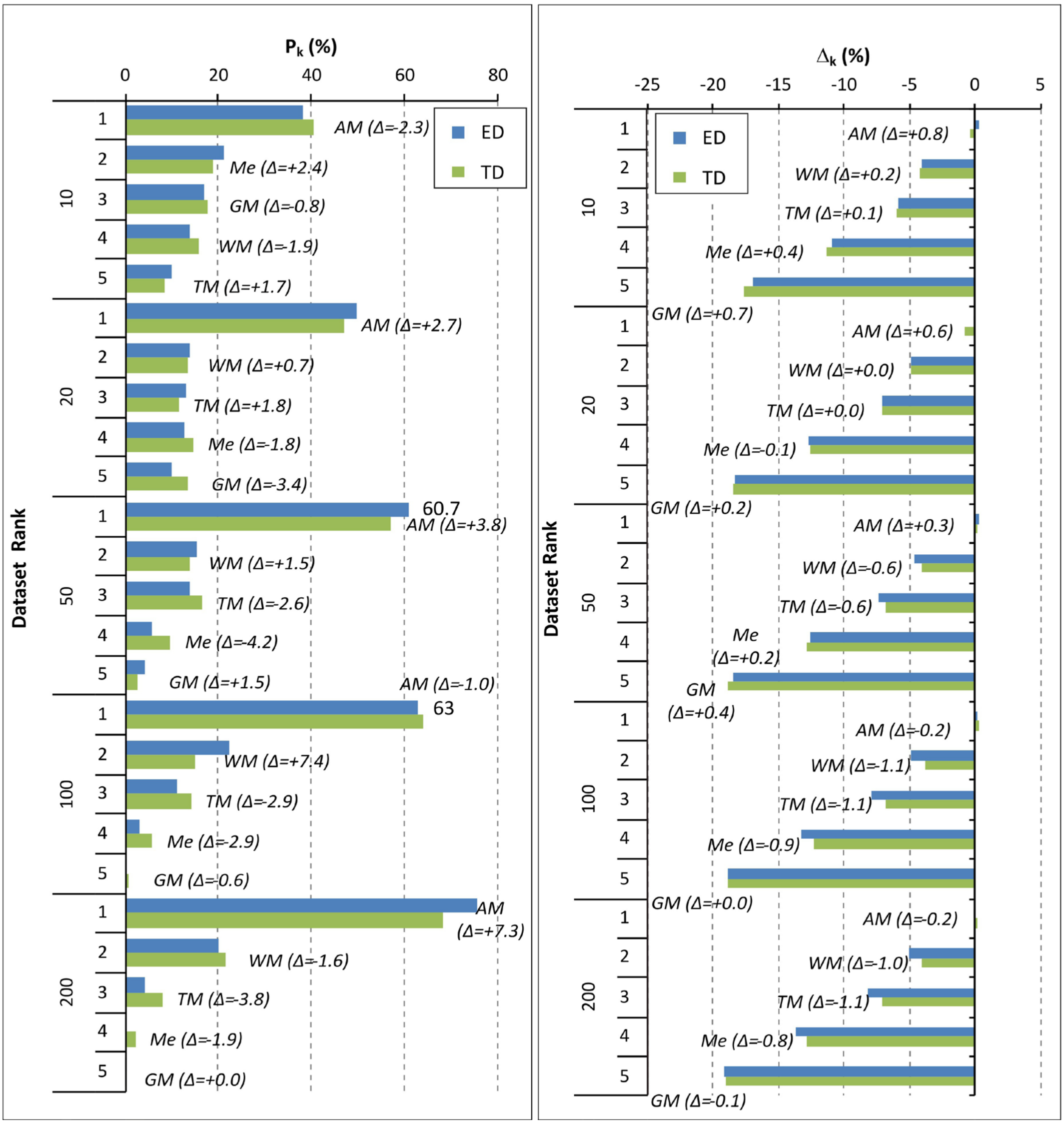


Fig. 4. Comparison of the ranking of estimators with respect to the reference population mean of Ni accumulation (*qNi*) based on 1000 series of random samples of different sizes generated from the empirical (*ED*) and the fitted theoretical gamma distributions (*TD*)

*AM* – Arithmetic Mean, *Me* – Median, *GM* – Geometric Mean, *TM* – Trimmed Mean, *WM* – Winsorized Mean; $n$ = 10, 20, 50, 100, 200 – sizes of random samples; Rankings are shown for: the percentage of cases in which each estimator yields the value closest to the reference population mean (left), and the average relative differences between estimator values and the reference population mean (right);

Δ(%) – difference between $P_k$ and $\Delta_k$ estimates based on random sample simulations from theoretical and empirical distributions of Ni accumulation

Rys. 4. Porównanie rankingu estymatorów pod względem bliskości oszacowań do referencyjnej średniej populacji zasobności Ni (*qNi*), oparte na 1000 seriach losowych prób o zróżnicowanej liczebności, generowanych z rozkładu empirycznego (*ED*) oraz dopasowanego rozkładu teoretycznego gamma (*TD*) mean for the random samples generated from the empirical and theoretical distributions are almost identical (Figure 4).

## Conclusions

The comparison of estimators with respect to their proximity to the reference population mean under conditions of right-skewed distributions of resource parameters with a large share of outliers was carried out using the Monte Carlo simulation method, demonstrated through the case study of Cu, Co, Ni and Pb accumulation in Cu-Ag deposits located in the LGCD area (Lubin–Głogów Copper District). The results should be interpreted as quantifying the divergence among estimates of the population mean of metal accumulation obtained using different measures of central tendency within the additive resource-estimation framework adopted here, rather than as establishing a universally preferred descriptive measure for skewed distributions. The results obtained may also be relevant to other deposit parameters that are defined by comparable empirical probability distribution structures, specifically those that show strong positive skewness, high variability, and the occurrence of numerous outliers, although this broader applicability should be verified in additional case studies.

Simulation experiments based on empirical distributions have shown that the estimator that most frequently yields values closest to the reference population mean is the arithmetic mean. Its advantage over the other measures considered of central tendency (the median, geometric mean, trimmed mean, and Winsorized mean) becomes more noticeable as the sample size increases.

Alternative measures of central tendency used in practice for estimating the population mean relevant to resource estimation, such as the median and, to a lesser extent, the geometric mean, may retain descriptive value and may occasionally provide acceptable results only for very small random samples. However, within the inferential framework adopted in this study, their performance as estimators of the population mean relevant to resource estimation is limited. Even in these cases, the frequency with which these measures occupy the top position in the ranking is clearly lower than that of the arithmetic mean. For large metal ore deposits, which are typically explored using a much larger number of samples (on the order of several dozen), the practical significance of the median and geometric mean appears limited. However, these measures may also be of some practical relevance for common mineral deposits (e.g., sand, gravel, clay), which are typically explored using a relatively small number of drill holes.

For larger data sets (more than 20 observations), the Winsorized mean and the trimmed mean provide a relatively good approximation of the reference population mean; in rankings of measures closest to the reference population mean, they typically occupy the second and third places, respectively, after the arithmetic mean.

From the perspective of the bias of the considered measures of central tendency, which in the case of the analyzed empirical distributions consists in underestimating the reference population mean, the results obtained from these distributions are consistent with the outcomes predicted by statistical theory for ideal theoretical probability distributions. The bias of the arithmetic mean, determined for random samples of different sizes, is close to zero. For the remaining measures of central tendency, the underestimation of the reference population mean increases in the following order: Winsorized mean, trimmed mean, median, and geometric mean.

It is also worth noting the persistent difficulty of satisfactorily approximating empirical distributions by theoretical models suitable for random sample generation. For the variables considered, only the distribution of Ni accumulation can be reliably approximated by a gamma distribution. These findings suggest that, in geological datasets of this type, empirical variability may be too complex to be represented reliably by a single simple theoretical distribution. Under such conditions, simulation based directly on empirical distributions provides the most methodologically defensible basis for comparing estimator performance.

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## DIVERGENCE AMONG ESTIMATORS OF THE POPULATION MEAN UNDER RIGHT-SKEWED DISTRIBUTIONS AND OUTLIERS – A CASE STUDY OF METAL ACCUMULATION IN Cu-Ag DEPOSITS OF THE LGCD

Keywords

metal accumulation, Cu-Ag deposits, right-skewed distributions,
outliers, Monte Carlo simulation, measures of central tendency

Abstract

The accumulation of metals in the sediment-hosted Cu-Ag deposits in the Lubin–Głogów Copper District (LGCD, SW Poland) are typically characterized by strongly right-skewed empirical distributions containing numerous outliers. This study quantifies the divergence among estimates of the population mean of metal accumulation obtained using five commonly used measures of central tendency, treated here as alternative estimators for the target mean relevant to additive resource estimation, under conditions of strong skewness and the presence of outliers.

Large empirical datasets of Cu, Co, Ni, and Pb accumulation values, obtained from ore-deposit sampling in the Rudna mine workings, were treated as reference populations, with their arithmetic means adopted as reference approximations of the population mean relevant to additive resource estimation. Monte Carlo simulations were used to generate 1,000 independent random samples of sizes from empirical distributions: $n$ = 10, 20, 50, 100 and 200.

Two evaluation criteria were applied:

- the frequency with which a given estimator produced values closest to the reference population mean,
- the mean relative deviation of the estimator from the reference population mean.

The results show that the arithmetic mean yields estimates with negligible bias (from −0.1% to +0.4%), whereas average relative underestimation of the reference population mean by the geometric mean reaches −16.6% for Cu, −29.4% for Co, −18.3% for Ni and −66.7% for Pb.

Within the adopted inferential framework, in which the target parameter is the population mean relevant to additive resource estimation and is approximated by the arithmetic mean of the reference population, the arithmetic mean most frequently yielded estimates closest to this reference value. Its advantage increased with sample size. The Winsorized and trimmed means showed intermediate performance, whereas the median and geometric mean tended to underestimate the reference population mean, particularly under strong skewness and in the presence of numerous outliers.

This conclusion applies specifically within the framework of additive resource estimation and does not necessarily extend to other descriptive uses of central tendency measures. Alternative measures such as the median or geometric mean may occasionally be useful for very small data sets; however, in the present study, they did not outperform the arithmetic mean. The study also highlights the limited usefulness of individual theoretical distributions in the description of empirical metal accumulation data in the LGCD, mainly due to the large number of outliers.

## ROZBIEŻNOŚCI MIĘDZY OSZACOWANIAMI ŚREDNIEJ POPULACJI W WARUNKACH ROZKŁADÓW PRAWOSTRONNIE ASYMETRYCZNYCH I OBECNOŚCI WARTOŚCI ANOMALNYCH – NA PRZYKŁADZIE ZASOBNOŚCI METALI W ZŁOŻACH Cu-Ag LGOM

Słowa kluczowe

zasobność metali, złoża Cu-Ag, rozkłady prawostronnie asymetryczne,
wartości anomalne, symulacja Monte Carlo, miary tendencji centralnej

Streszczenie

Zasobność metali w złożach Cu-Ag typu sedymentacyjnego w rejonie Lubina–Głogowa (LGOM, SW Polska) charakteryzuje się zazwyczaj silnie prawostronnie skośnymi rozkładami empirycznymi, zawierającymi liczne wartości odstające. Takie cechy komplikują zarówno dopasowanie rozkładów teoretycznych do danych empirycznych, jak i wybór właściwej miary tendencji centralnej – kluczowego parametru wykorzystywanego w szacowaniu zasobów. W prezentowanych wynikach badań ilościowo określono rozbieżności między oszacowaniami średniej w populacji zasobności metali, istotnej z punktu widzenia addytywnego szacowania zasobów, uzyskiwanymi przy użyciu pięciu powszechnie stosowanych miar tendencji centralnej – średniej arytmetycznej, mediany, średniej geometrycznej, średniej obciętej oraz średniej winsorowskiej – w warunkach silnej skośności i obecności obserwacji ekstremalnie wysokich.

Ogromne zbiory danych empirycznych pomiarów zasobności Cu, Co, Ni i Pb, uzyskane w wyniku opróbowania złoża w wyrobiskach górniczych kopalni Rudna, potraktowano jako populacje referencyjne, a ich średnie arytmetyczne przyjęto jako referencyjne przybliżenia średniej populacji istotnej dla addytywnego szacowania zasobów. Symulacje Monte Carlo wykorzystano do wygenerowania z rozkładów empirycznych zasobności każdego z metali 1000 niezależnych losowych próbek o rozmiarach $n$ = 10, 20, 50, 100 i 200.

Zastosowano dwa kryteria oceny:

- częstość, z jaką dany estymator dostarczał wartości najbliższej referencyjnej średniej populacji,
- średnie względne odchylenie estymatora od tej wartości.

Średnia arytmetyczna konsekwentnie przewyższała pozostałe miary, a jej przewaga zwiększała się wraz z liczebnością próby. Średnia winsorowska zajmowała drugą pozycję, następnie średnia obcięta, natomiast mediana i średnia geometryczna były najmniej trafnymi estymatorami i systematycznie zaniżały referencyjną średnią populacji – często w sposób znaczny przy skrajnie silnej skośności i obecności bardzo licznych wartości ekstremalnie wysokich (szczególnie w przypadku zasobności Pb). Wyniki symulacji pokazały, że stosowanie średniej arytmetycznej skutkuje zaniedbywalnie małym błędem oszacowania referencyjnej średniej populacji (od –0,1% do 0,4%) podczas gdy przeciętne, relatywne niedoszacowanie tej wartości przez średnią geometryczną wynosi: –16,6% dla zasobności Cu, –29,4% dla zasobności Co, –18,3% dla zasobności Ni i –66,7% dla zasobności Pb.

Pomimo silnie skośnych rozkładów zasobności i licznych wartości odstających średnia arytmetyczna najczęściej dawała oszacowania najbliższe referencyjnej średniej populacyjnej zasobności metali przyjętej w tym badaniu. Wniosek ten odnosi się do ram addytywnego szacowania i nie musi automatycznie rozciągać się na inne opisowe zastosowania miar tendencji centralnej. Alternatywne

miary, takie jak mediana lub średnia geometryczna, mogą okazać się przydatne w przypadku bardzo małych zbiorów danych, jednak w niniejszym badaniu nie przewyższały średniej arytmetycznej. Badanie podkreśla również ograniczoną przydatność indywidualnych rozkładów teoretycznych do aproksymacji zasobności metali w złożach LGOM.